**THE UNIVERSITY OF QUEENSLAND**
**SCHOOL OF MATHEMATICS AND PHYSICS**

**Mid-semester Examination 2017**
**Thursday April 13, 2017**
**Room 407, Physics Annexe**

# PHYS4030/7033: Condensed matter physics

(SCIENCE)

TIME: Fifty minutes for working.
Five minutes for perusal (no writing) before examination begins.

There are THREE (3) questions
Answer any TWO (2) questions
Each question and each part of each question is worth equal marks.

No formula sheet is provided. Instead you may bring your own formula sheet which is two sides of a single A4 sheet of paper. It may contain no more than 20 equations. You must hand in your formula sheet with your exam book at the end of the exam.

Approved calculators allowed.

1.

Consider an electron in a **weak** periodic potential in one dimension.

$$U(x) = \sum_{n=0}^{\infty} U_{nG} \cos(Gx)$$

$G$ is the magnitude of the primitive vector for the reciprocal lattice. $U_0, U_G, U_{2G}, U_{3G}, \cdots$ denote the Fourier components of the periodic potential.

From first-order degenerate perturbation theory it can be shown that near the centre of the first Brillouin zone the dependence of the **second and third** lowest energy eigenvalues $\epsilon(k)$ on the Bloch wavevector $k$ is given by

$$\epsilon(k) = \frac{1}{2}(\epsilon^0(k+G) + \epsilon^0(k-G)) \pm \left[|U_{2G}|^2 + [\frac{1}{2}(\epsilon^0(k+G) - \epsilon^0(k-G))]^2\right]^{\frac{1}{2}}$$

where $\epsilon^0(k) = (\hbar k)^2/(2m_e)$ is the energy of free electrons, and $m_e$ is the free electron mass.

*Caution:* this is a different formula from the one you encountered previously in an assignment problem because we are discussing different bands.

(a)

Sketch the band structure for these two bands in the first Brillouin zone.

(b)

Use the Laue equation ($\vec{k'} = \vec{k} + \vec{K}$) to give an simple explaination of the physics that the above formula for the two bands is describing.

(c)

Suppose that there are on average 3.8 electrons per lattice site. Explain why the system is a metal and not an insulator. Are the dominant charge carriers electrons or holes?

(d)

For the system in (c) find the Fermi wavevectors and show them and the Fermi energy on your band structure diagram.

(e)

Use the expression for $\epsilon(k)$ to show that the effective mass $m^*$ of holes near the top of the second band is

$$\frac{m^*}{m_e} = \frac{|U_{2G}|}{2\ \epsilon^0(G)},$$

provided the potential is weak.

2.

Consider a non-interacting gas of electrons in two dimensions. Let $n$ denote the density of electrons per unit area and $m^*$ the effective mass of the electrons.

(a)

Suppose that the system is a square with sides of length $L$. For periodic boundary conditions, what are the allowed values of the wave vectors $\vec{k}$ which label the possible quantum states of the electrons?

(b)

Show that the Fermi wave vector $k_F$ is related to the density $n$ by

$$n = \frac{k_F^2}{2\pi}$$

(c)

If the density of electrons is $n = 5 \times 10^{11}/\mathrm{cm}^2$ in a AlGaAs semiconductor heterostructure with effective mass, $m^* = 0.07 m_e$ (where $m_e$ is the free electron mass), evaluate the magnitude of the Fermi wave vector.

(d)

One can add electrical gates in a periodic pattern to induce a one-dimensional potential $V(x, y) = V_0 \cos(2\pi x/a)$ experienced by the electrons in the two-dimensional electron gas. What is the minimum value of $a$ that is allowed if the potential is to change the shape of the Fermi surface? Illustrate your argument by drawing a picture of the associated First Brillouin Zone (FBZ) on top of a picture of the circular Fermi surface that is present in the absence of the potential.

(e)

For the AlGaAs semiconductor heterostructure discussed above give a rough estimate of the magnitude of the periodic potential $V_0$ (in eV) that would be required for the new Fermi surface to contain hole orbits.

3.

Maxwell's equations of electromagnetism in a non-magnetic medium with dielectric constant $\epsilon$ are:

$$\nabla \cdot \vec{E} = \rho/\epsilon$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Suppose there is an oscillating electric field $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}, \omega) \exp(i\omega t)$ with angular frequency $\omega$ which drives an oscillating electric current given by the frequency dependent Ohm's law

$$\vec{J}(\omega) = \sigma(\omega)\vec{E}(\omega)$$

(a)

Use the identity

$$\nabla \times \nabla \times \vec{A} = -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A})$$

to derive a wave equation for $\vec{E}(\vec{r}, \omega)$, where the speed of propagation of the radiation $v$ is related to the speed of light in a vacuum $c = 1/\sqrt{\mu_0 \epsilon_0}$ and given by $v^2 = 1/(\mu_0 \epsilon(\omega))$ with the frequency-dependent dielectric constant and conductivity related by

$$\epsilon(\omega) = \epsilon_0 + \frac{i}{\omega}\sigma(\omega)$$

(b)

Show that in the Drude model for frequencies much larger than the scattering rate the frequency-dependent dielectric constant is given by

$$\epsilon(\omega) = \epsilon_0 \left(1 - (\frac{\omega_p}{\omega})^2\right),$$

where the plasma frequency $\omega_p$ is given by $\omega_p^2 = ne^2/m\epsilon_0$.

(c)

Explain why the plasma frequency determines a qualitative difference in the behavior of the propagation of electromagnetic radiation in a metal.

(d)

The mass density of sodium (atomic weight, A=23) at a temperature of 295 Kelvin is 1.0 grams/cm$^3$. Each atom contributes one valence electron. Calculate the number density of valence electrons.

(e)

Evaluate the wavelength of electromagnetic radiation below which sodium becomes transparent. Compare your value to the experimental value for sodium of 2100 $\AA$.